Existence of Competitive Equilibrium

Econ 3030 Fall 2025

Lecture 23

Outline

1. Existence of a competitive (Walrasian) equilibirum

From Last Class

Definition

The market excess demand correspondence $z : \mathbb{R}^L_+ \to \mathbb{R}^L$ is:

$$z(\mathbf{p}) = \sum_{i=1}^{I} (\mathbf{x}_i^* - \omega_i) - \sum_{j=1}^{J} \mathbf{y}_j^*$$

where $\mathbf{y}_j^* \in y_j^*(\mathbf{p})$ for each $j = 1, ..., J$, and $\mathbf{x}_i^* \in x_i^*(\mathbf{p})$ for each $i = 1, ..., I$.

REMARK

An equilibrium price vector $\mathbf{p}^* > 0$ satisfies:

$$z_l(\mathbf{p}^*) \leq 0 \text{ for all } l, \quad \text{and} \quad \text{whenever } z_l(\mathbf{p}^*) < 0 \text{ then } p_l^* = 0$$

Walras' Law

If preferences are locally non satiated, for any price vector $\mathbf{p} \in \mathbb{R}^L_+$:

$$\mathbf{p} \cdot z(\mathbf{p}) = 0$$

Equilibrium As A Fixed Point

- An equilibrium price vector $\mathbf{p}^*$ must satisfy:

$$z_l(\mathbf{p}^*) \leq 0 \text{ for all } l, \qquad \text{and} \quad \text{whenever } z_l(\mathbf{p}^*) < 0 \text{ then } p_l^* = 0$$

A useful observation

- Let the function $g : \mathbb{R}^L \to \mathbb{R}^L$ be defined by

$$g_l(\mathbf{p}) = \max\{p_l + z_l(\mathbf{p}), 0\} \qquad \text{for } l = 1, 2, .., L$$

- CLAIM: An equilibrium is a $\mathbf{p}^* \geq 0$ such that

$$g(\mathbf{p}^*) = \mathbf{p}^* \qquad \text{or} \qquad g_l(\mathbf{p}^*) = p_l^* \text{ for all } l$$

Proof: At an equilibrium price vector p^*:

1. either $p_l^* = g_l(\mathbf{p}^*) = 0$, and thus either $z_l(\mathbf{p}^*) < 0$ or $z_l(\mathbf{p}^*) = 0$
2. or $p_l^* = g_l(\mathbf{p}^*) \neq 0$, and thus $p_l^* + z_l(\mathbf{p}^*) = p_l^*$ which implies $z_l(\mathbf{p}^*) = 0$.

- In both cases we have an equilibrium thus establishing the claim.

Equilibrium As A Fixed Point

- Let the function $g : \mathbb{R}^L \to \mathbb{R}^L$ be defined by

$$g_l\left(\mathbf{p}\right) = \max\left\{p_l + z_l\left(\mathbf{p}\right), 0\right\} \qquad \text{for } l = 1, 2, .., L$$

Summary

- An equilibirum exists if there exists a $\mathbf{p}^*$ such that

$$g\left(\mathbf{p}^*\right) = \mathbf{p}^*$$

- This is a fixed point: we want to show that $g(\mathbf{p})$ must have a fixed point.

- We need a theorem that gives conditions for functions to have a fixed point.

Brouwer's Fixed Point Theorem

Theorem (Brouwer's Fixed Point Theorem)

If $X \subseteq \mathbb{R}^L$ is convex and compact and the function $f : X \to X$ is continuous, then there exists an $\mathbf{x} \in X$ such that $f(\mathbf{x}) = \mathbf{x}$ (that is, f has a fixed point).

- Counterexamples:
  - $X = (0, 1]$ and $f(x) = \frac{x}{2}$ (X must be closed)
  - $X = [0, \infty)$ and $f(x) = x + 1$ (X must be bounded)
  - $X = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$ and $f(\mathbf{x}) = -\mathbf{x}$ (X must be convex)
  - $X = [0, 1]$ and $f(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 0.5 \\ 0 & \text{if} \quad 0.5 < x \leq 1 \end{cases}$ (f must be continuous)
- In general, we will want to allow for correspondences (aggregate excess demand may not be a function);
  - fortunately, there is similar theorem for correspondences (Kakutani's fixed point theorem).

Brouwer's Theorem and Existence

Theorem

A continuous $f : S \to S$, where $S \subseteq \mathbb{R}^L$ is convex and compact, has a fixed point.

How can we use this theorem to show that an equilibirum exist?

Let $g : \mathbb{R}^L \to \mathbb{R}^L$ be defined by: $g_l(\mathbf{p}) = \max\{p_l + z_l(\mathbf{p}), 0\}$ for $l = 1, 2, .., L$. Can we apply Brower?

1. $g(\cdot)$ needs to be continuous.
   Assume preferences that yield continuity of excess demand. ✓
2. $g(\cdot)$ needs to be a function.
   Assume strictly convex preferences (there is also a theorem for correspondences). ✓
3. Need domain and range to be the same convex and compact subset of $\mathbb{R}^L$.
   Normalize prices so that the domain is $\Delta^{L-1} = \{\mathbf{p} \in \mathbb{R}^L_+ : \sum_{l=1}^{L} p_l = 1\}$, and then divide $g(\cdot)$ by the sum of its elements so that the range is also Δ^{L-1}. ✓
4. $g(\cdot)$ must be well defined even if some prices are zero.
   If preferences are monotone, excess demand can blow up. But we need monotonicity for other properties. Assume this away (for now). ☒

'Easy' Existence Theorem

Theorem

Assume that aggregate excess demand $z : \Delta^{L-1} \to \mathbb{R}^L$ *is a continuous function such that* $\mathbf{p} \cdot z(\mathbf{p}) = 0$ *for all* $\mathbf{p}$*. Then, there exists* $\mathbf{p}^* \in \Delta^{L-1}$ *such that* $z(\mathbf{p}^*) \leq 0$.

- An equilibrium exists if excess demand is a continuous function satisfying Walras' Law.

Remark

- We should prove existence from assumptions on the primitives of the economy (technology, preferences, and endowments), not on excess demand.
  - Although many of the assumptions can be derived from preferences, the one that excess demand is well defined and continuous over its entire domain cannot (need to rule out zero prices).

The proof goes as follows

1. Define a function of excess demand so that...
2. ... Brouwer's fixed point theorem applies, and a fixed point of this function exists.
3. Complete the proof by showing this fixed point is an equilibrium.

'Easy' Existence Proof: Step 1a

Let $g : \Delta^{L-1} \to \mathbb{R}^L$ *be defined by*

$$g_l(\mathbf{p}) = \max\{p_l + z_l(\mathbf{p}), 0\} \quad \textit{with } l = 1, 2, .., L.$$

Remark

Claim: $g(\mathbf{p}) \neq 0$.

- By definition, $g_l(\mathbf{p}) \geq p_l + z_l(\mathbf{p})$, and multiplying both sides by p_l we have:
$$p_l g_l(\mathbf{p}) \geq p_l(p_l + z_l(\mathbf{p}))$$
- Summing over l:
$$\begin{aligned} \mathbf{p} \cdot g(\mathbf{p}) &\geq \mathbf{p} \cdot (\mathbf{p} + z(\mathbf{p})) \\ &= \mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot z(\mathbf{p}) \\ &= \mathbf{p} \cdot \mathbf{p} + \underbrace{0}_{\text{by Walras' Law}} \quad \underbrace{> 0}_{\text{because } p \in \Delta^{L-1}} \end{aligned}$$
- $g_l(\mathbf{p})$ cannot be equal to zero for all l (some good must have a positive price)
- Thus $\max\{p_l + z_l(\mathbf{p}), 0\} > 0$ for at least one l.

'Easy' Existence Proof: Step 1b

Define $h : \Delta^{L-1} \rightarrow \Delta^{L-1}$ *as*

$$h(\mathbf{p}) = \frac{g(\mathbf{p})}{\sum_{l=1}^{L} g_l(\mathbf{p})}$$

- $h(\cdot)$ is well defined.
- $h(\cdot)$ is continuous because $z(\cdot)$ is continuous and thus $g(\cdot)$ is continuous.
- $h(\cdot)$ maps from Δ^{L-1}, a convex and compact set, to itself.

Therefore

$h(\cdot)$ is a continuous function from a compact and convex set to itself; by Brouwer's theorem, it has a fixed point.

'Easy' Existence Proof: Step 2

By Brouwer's fixed point theorem, there exists a p^* such that

$$\mathbf{p}^* = h(\mathbf{p}^*) = \frac{g(\mathbf{p}^*)}{\sum_{l=1}^{L} g_l(\mathbf{p}^*)}$$

- Rewrite this as

$$g(\mathbf{p}^*) = \left(\sum_{l=1}^{L} g_l(\mathbf{p}^*) \right) \mathbf{p}^* = \gamma \mathbf{p}^*$$

  for some real number γ.
- Observe that $\gamma \neq 0$ because $g(\mathbf{p}) \neq 0$ from previous slide.
- Next, we show that $\gamma = 1$.

'Easy' Existence Proof: Step 3a

Claim: *for each* $l = 1, .., L$: $\quad p_l^* g_l(\mathbf{p}^*) = p_l^*(p_l^* + z_l(\mathbf{p}^*))$

- *This is easy to show:*

$$\text{if } g_l(\mathbf{p}^*) \neq \mathbf{p}_l^* + z_l(\mathbf{p}^*) \overbrace{\Rightarrow}^{\text{by definition}} g_l(\mathbf{p}^*) = 0 \overbrace{\Rightarrow}^{\text{by } \gamma \neq 0} p_l^* = 0$$

- *Thus: either* $g_l(\mathbf{p}^*) = p_l^* + z_l(\mathbf{p}^*)$ *or* $p_l^* = 0$*; in both cases the claim holds.*

- Summing over l, we obtain:

$$\mathbf{p}^* \cdot g(\mathbf{p}^*) = \mathbf{p}^* \cdot (\mathbf{p}^* + z(\mathbf{p}^*)) = \mathbf{p}^* \cdot \mathbf{p}^* + \mathbf{p}^* \cdot z(\mathbf{p}^*) = \mathbf{p}^* \cdot \mathbf{p}^* + \overbrace{0}^{\text{by Walras' Law}}$$

- By the existsence of a fixed point (last slide), we know that $g(\mathbf{p}^*) = \mathbf{p}^*\gamma$; taking the dot product with $\mathbf{p}^*$ on both sides:

$$\mathbf{p}^* \cdot g(\mathbf{p}^*) = \mathbf{p}^* \cdot \mathbf{p}^*\gamma$$

- Since we have just shown that $\mathbf{p}^* \cdot g(\mathbf{p}^*) = \mathbf{p}^* \cdot \mathbf{p}^*$, we have

$$\mathbf{p}^* \cdot \mathbf{p}^* = \mathbf{p}^* \cdot \mathbf{p}^*\gamma$$

  and thus $\gamma = 1$ as desired.

'Easy' Existence Proof: Step 3b

Summary

By the fixed point theorem, $g(\mathbf{p}^*) = \mathbf{p}^*\gamma$, and since $\gamma = 1$: $g(\mathbf{p}^*) = \mathbf{p}^*$.

The fixed point is an equilibrium

Claim: $\mathbf{p}^*$ is an equilibrium.

Proof.

- The equation above implies:

$$p_l^* = g_l(\mathbf{p}^*) = \underbrace{\max\{p_l^* + z_l(\mathbf{p}^*), 0\}}_{\text{by definition of } g(\cdot)} \qquad \text{for } l = 1, 2, .., L$$

- Therefore: $z_l(\mathbf{p}^*) \leq 0 \qquad \text{for } l = 1, 2, .., L$
  - If not, there exists some k such that $z_k(\mathbf{p}^*) > 0$
  - and $p_k^* = \max\{p_k^* + z_k(\mathbf{p}^*), 0\} = p_k^* + z_k(\mathbf{p}^*)$
    an impossibility.
- Since $\mathbf{p}^*$ is an equilibirum if and only if $z(\mathbf{p}^*) \leq 0$, we are done. □

What Took So Long

- In 1950, John Nash proved existence of the (Nash) equilibrium of a game.
- This was the breakthrough needed by Arrow, Debreu, and McKenzie to prove existence of a competitive equilibrium shortly thereafter.
- A "game" is a situation in which an individual's payoff may depend on others' choices.
  - We have not looked at games
- Since the connection between the two is interesting, we will take a detour into game theory and define a game formally.

Games and Nash Equilibrium

- A game is defined by describing choices and payoffs for each player.
- each $i = 1, ..., I$ chooses a strategy in the set S_i;
  - let $S = S_1 \times S_2 \times .. \times S_I$, and $\mathbf{s} = (s_1, .., s_I) \in S$ is a strategy profile (an action for each player);
- $u_i(s_1, .., s_i, .., s_I) = u_i(s)$ is player i's payoff function from $\mathbf{s} = (s_1, .., s_i, .., s_I)$.
  - Rewrite $u_i(s_1, .., s_i, .., s_I)$ as $u_i(s_i, \mathbf{s}_{-i})$ where $\mathbf{s}_{-i} = s_1, .., s_{i-1}, s_{i+1}, .., s_I$
    - i takes the choices of other players as given
- A Nash equilibrium is a strategy profile such that: each player maximizes her payoff given the other players' strategies.
- Thus, a Nash equilibrium is an $\mathbf{s}^*$ such that for all i

$$u_i(s_i^*, \mathbf{s}_{-i}^*) \geq u_i(t, \mathbf{s}_{-i}^*) \qquad \text{for any } t \in S_i.$$

- Nash proved that such a strategy profile exists in any game when (i) each strategy set is compact and convex, and (ii) each payoff functions is strictly quasi-concave and continuous.
- How? Using Brower's Theorem.

Nash's Existence Theorem

A Nash equilibirum is a fixed point

- For each i, let $BR_i : S_{-i} \longrightarrow S_i$ be defined as follows:

$$BR_i\left(\mathbf{s}_{-i}\right) = \{s_i \in S_i : \; u_i\left(s_i, \mathbf{s}_{-i}\right) \geq u_i\left(t, \mathbf{s}_{-i}\right) \quad \forall t \in S_i\}$$

  - this function describes i's 'best response' to the other players strategy $\mathbf{s}_{-i}$.

- Define the mapping $BR : S \longrightarrow S$ as

$$BR\left(\mathbf{s}\right) = BR_1\left(\mathbf{s}_{-1}\right) \times ... \times BR_I\left(\mathbf{s}_{-I}\right)$$

- $\mathbf{s}^*$ is a Nash equilibrium if and only if $\mathbf{s}^* = BR\left(\mathbf{s}^*\right)$: a fixed point.
  - In a Nash equilibrium every player chooses a best response.

- Assumptions (i) and (ii) from the previous slide are enough to use Brouwer's Theorem.
  - we need S closed, bounded, and convex; so, assume each S_i has those proerties.
  - we need $BR\left(\mathbf{s}\right)$ to be a continuous function; so, assume each $u_i(\mathbf{s})$ is continuous and strictly quasi-concave, so that each $BR_i\left(\mathbf{s}\right)$ is a continuous function.
- Under these assumption, every game has a Nash equilibrium.

From Nash to Walras

- Arrow, Debreu, and McKenzie used Nash's result to solve the existence problem.

Build a 'game' that describes a Walrasian (competitive) equilibirum

- The "players" are: consumers, firms, and a 'Price Player'.
- The price player chooses a price vector $\mathbf{p} \in \Delta^{L-1}$.
- Consumer i's payoff function is $u_i(\mathbf{x}_i)$ (i's utility function): i's best response is an element of Walrasian demand $\mathbf{x}_i^*(\mathbf{p})$.
- Firm j's payoff function is $\mathbf{p} \cdot \mathbf{y}_j$: j's best response is an element of the supply $\mathbf{y}_j^*(\mathbf{p})$.
- The Price Player's payoff is the value of the aggregate excess demand:

$$u_{PP}(\mathbf{p}, \mathbf{x}) = \mathbf{p} \cdot \mathbf{z} = \mathbf{p} \cdot \left(\sum_{i=1}^{I} (\mathbf{x}_i^* - \omega_i) - \sum_{j=1}^{J} \mathbf{y}_j^* \right)$$

  the best response is a price that maximizes $\mathbf{p} \cdot \mathbf{z}$ (the value of that excess demand).
- A Nash equilibrium of this game: prices, consumption, and production such that all players choose a best response.

From Nash to Walras

A Nash equilibirum is a fixed point

- By Nash's theorem, there exist a $\mathbf{p}^*$, $\mathbf{x}_i^*$ for each i, and $\mathbf{y}_j^*$ for each j, such that
  - each $\mathbf{y}_j^*$ maximizes profits given $\mathbf{p}^*$,
  - each $\mathbf{x}_i^*$ maximizes individuals' utility given $\mathbf{p}^*$ and $\mathbf{y}_j^*$, and
  - $\mathbf{p}^*$ maximizes the value of aggregate excess demand (given $\mathbf{x}_i^*$ and $\mathbf{y}_j^*$).

Claim: this is a Walrasian equilibrium

- Why is $\mathbf{p}^*$ a competitive equilibrium?
  - Walras' Law implies $\mathbf{p}^* \cdot z(\mathbf{p}^*) = 0$.
  - The price player maximization implies $\mathbf{p}^* \cdot z(\mathbf{p}^*) \geq \mathbf{p} \cdot z(\mathbf{p})$ for all $\mathbf{p} \in \Delta^{L-1}$.
  - These together imply $z(\mathbf{p}^*) \leq 0$ (make sure you convince yourself of this).
  - We already proved that $z(\mathbf{p}^*) \leq 0$ implies $\mathbf{p}^*$ is an equilibrium.

Remarks

- To use Brouwer's theorem, best responses must be functions.
  - even if aggregate excess demand is a function, $BR_{PP}(\cdot)$ can be multi-valued.
- We need fixed point existence for correspondences: Kakutani's theorem.

An General Existence Theorem

Theorem

Suppose an economy satisfies the following properties.

1. *For each i:* $X_i \subset \mathbb{R}^L$ *is closed and convex*
   $\succsim_i$ *satisfies local non-satiation, and convexity*
   $\omega_i \geq \widehat{x}_i$ *for some* $\widehat{x}_i \in X_i$.
2. *For each j:* $Y_j \subset \mathbb{R}^L$ *is closed, convex,*
   includes the origin, and satisfies free-disposal.
3. *The set of feasible allocations is compact.*

Then a Walrasian quasi-equilibrium exists (if $\omega_i \gg \widehat{x}_i$ *for all i then an equilibrium exists).*

- Issues a proof needs to take care of.
  - When some prices are zero some individual's demand can explode.
    - This is because the budget set is not compact.
  - Demand (and therefore excess demand) is not necessarily continuous at zero prices.
    - Think about how could this happen (see problem set 10).
  - We need all prices to be strictly positive.

Proving a Not So Simple Existence Theorem

- This is a sketch of the proof for existence of a competitive equilibrium.

1. Truncate the economy, so that all choices must belong to a compact set.
2. Construct a 'game' with $I + J + 1$ players: consumers, firms, and the 'price player'.
3. Show that each player's best response is a non-empty, convex, and upper hemi-continuous correspondence.
4. Hence the 'product' best-response correspondence that describes this game inherits those properties.
5. Use Kakutani's fixed point theorem to show this correspondence has a fixed point.
6. This fixed point is an equilibrium of the truncated economy.
7. Prove that the truncation does not matter: an equilibrium of the truncated economy is a quasi-equilibrium of the whole economy.
8. DONE.

- Some of the tricky issues have to do with getting strictly positive prices so that the correspondences are upper hemi-continuous. See a book for the proof.

Next Class

- Uncertainty